

DYNAMICAL DEGREE AND ARITHMETIC DEGREE OF ENDOMORPHISMS ON PRODUCT VARIETIES

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ABSTRACT. For a dominant rational self-map on a smooth projective variety defined over a number field, Shu Kawaguchi and Joseph H. Silverman conjectured that the (first) dynamical degree is equal to the arithmetic degree at a rational point whose forward orbit is well-defined and Zariski dense. We give some examples of self-maps on product varieties and rational points on them for which the Kawaguchi-Silverman conjecture holds.

1. INTRODUCTION

Let X be a smooth projective variety defined over a number field k , and $f: X \dashrightarrow X$ a dominant rational self-map defined over k . Let $I_f \subset X$ be the indeterminacy locus of f . Let $X_f(\bar{k})$ be the set of \bar{k} -rational points P on X such that $f^n(P) \notin I_f$ for every n . For a \bar{k} -rational point $P \in X_f(\bar{k})$, its *forward f -orbit* is defined by $\mathcal{O}_f(P) := \{f^n(P) : n \geq 0\}$.

Let H be an ample divisor on X defined over k . The (first) *dynamical degree* of f is defined by

$$\delta_f := \lim_{n \rightarrow \infty} \deg(((f^n)^*H) \cdot H^{\dim X - 1})^{1/n}.$$

The *arithmetic degree* of f at a \bar{k} -rational point $P \in X_f(\bar{k})$ is defined by

$$\alpha_f(P) := \lim_{n \rightarrow \infty} h_H^+(f^n(P))^{1/n}$$

if the limit on the right hand side exists. Here, $h_H: X(\bar{k}) \rightarrow [0, \infty)$ is the (absolute logarithmic) Weil height function associated with H , and we put $h_H^+ := \max\{h_H, 1\}$.

Shu Kawaguchi and Joseph H. Silverman formulated the following conjecture.

Conjecture 1.1 (Kawaguchi-Silverman conjecture (see [18, Conjecture 6])). *For every \bar{k} -rational point $P \in X_f(\bar{k})$, the arithmetic degree $\alpha_f(P)$ is defined. Moreover, if the forward f -orbit $\mathcal{O}_f(P)$ is Zariski dense in X , the arithmetic degree $\alpha_f(P)$ is equal to the dynamical degree δ_f , i.e., we have*

$$\alpha_f(P) = \delta_f.$$

The existence of the limit defining the arithmetic degree when f is a dominant *endomorphism* (i.e., f is defined everywhere) is proved in [16]. But in general, the convergence is not known. It seems difficult to prove Conjecture 1.1 in full generality.

The following variant of Conjecture 1.1 is also studied by Kawaguchi and Silverman in [17].

Conjecture 1.2 (see [17, Theorem 3]). *The set*

$$\{P \in X_f(\bar{k}) : \alpha_f(P) \text{ is defined and equal to } \delta_f\}$$

contains a Zariski dense set of points having disjoint orbits.

The aim of this paper is to give examples of endomorphisms on product varieties and rational points on them for which Conjecture 1.1 or Conjecture 1.2 is true.

We prove Conjecture 1.1 in the following situations.

Theorem 1.3. *For $i = 1, 2, \dots, n$, let X_i be a smooth projective variety defined over a number field k . Assume that each X_i satisfies at least one of the following conditions:*

- *the first Betti number of $X_i(\mathbb{C})$ is zero and the Néron-Severi group of $X_i \otimes_k \bar{k}$ has rank one,*
- *X_i is an abelian variety,*
- *X_i is an Enriques surface, or*
- *X_i is a K3 surface.*

Then Conjecture 1.1 is true for any endomorphism $f: \prod_{i=1}^n X_i \rightarrow \prod_{i=1}^n X_i$ defined over k .

We also prove that, when one of the direct factors is of general type, any endomorphism on the product variety does not admit Zariski dense forward orbit. Thus, Conjecture 1.1 is obviously true for such endomorphisms.

Theorem 1.4. *Let X and Y be smooth projective varieties of dimension ≥ 1 defined over a subfield $k \subset \mathbb{C}$, and $f: X \times Y \rightarrow X \times Y$ an endomorphism defined over k . Assume that at least one of X or Y is of general type. Then, for every \bar{k} -rational point $P \in (X \times Y)(\bar{k})$, the forward f -orbit $\mathcal{O}_f(P)$ is not Zariski dense in $X \times Y$.*

We prove Conjecture 1.2 for certain rational self-maps on the projective space \mathbb{P}^N .

Theorem 1.5. *Let $f: \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ be a dominant rational self-map defined over a number field k whose restriction to \mathbb{A}^N is a morphism written as*

$$f(x_1, x_2, \dots, x_N) = (f_1(x_1, x_2, \dots, x_N), f_2(x_2, x_3, \dots, x_N), \dots, f_N(x_N))$$

for some polynomials $f_i \in k[x_i, x_{i+1}, \dots, x_N]$ for $i = 1, 2, \dots, N$. Assume that at least one of the following conditions is satisfied:

- *$\deg_{x_i} f_i > \deg_{x_{i+1}} f_{i+1}$ for every i , or*
- *$N = 2$.*

Then the set

$$\{P \in \mathbb{A}^N(\bar{k}) : \alpha_f(P) \text{ is defined and equal to } \delta_f\}$$

contains a Zariski dense set of points having disjoint orbits.

Remark 1.6. Kawaguchi and Silverman proved Conjecture 1.1 in the following cases (for details, see [16], [17], [21], [22]).

- ([17, Theorem 2 (a)]) f is an endomorphism and the Néron-Severi group of $X \otimes_k \bar{k}$ has rank one.
- ([17, Theorem 2 (b)]) f is the extension to \mathbb{P}^N of a regular affine automorphism on \mathbb{A}^N .
- ([17, Theorem 2 (c)]) X is a smooth projective surface and f is an automorphism on X .
- ([21, Proposition 19]) f is the extension to \mathbb{P}^N of a monomial endomorphism on \mathbb{G}_m^N and $P \in \mathbb{G}_m^N(\bar{k})$.
- ([16, Corollary 31], [22, Theorem 2]) X is an abelian variety. Note that any rational map between abelian varieties is automatically a morphism, and can be written as the composition of a homomorphism of abelian varieties and the translation by a point.

Remark 1.7. Kawaguchi and Silverman proved Conjecture 1.2 in the following cases (for details, see [17]).

- ([17, Theorem 3 (a)]) f is an algebraically stable dominant rational self-map on \mathbb{P}^2 whose restriction to \mathbb{A}^2 is an affine morphism.
- ([17, Theorem 3 (b)]) f is a dominant rational self-map on \mathbb{P}^2 with $\deg(f) = 2$ whose restriction to \mathbb{A}^2 is an affine morphism.
- ([17, Lemma 21]) f is a dominant rational self-map on \mathbb{P}^N with $\delta_f > 1$ such that its restriction to \mathbb{A}^N is an affine morphism, and there exists a \bar{k} -rational point $Q_0 \in \mathbb{P}^N(\bar{k})$ lying on the hyperplane at infinity satisfying that f^m is defined at Q_0 and $f^m(Q_0) = Q_0$ for some $m \geq 1$.

Notation. The base field k is always a number field or a subfield of \mathbb{C} . A variety defined over k means a scheme of finite type over $\text{Spec } k$ which is geometrically integral. An endomorphism on a variety X means a morphism from X to itself. For a smooth projective variety defined over a number field k , $b_1(X) := \dim_{\mathbb{Q}} H^1(X(\mathbb{C}), \mathbb{Q})$ denotes the first Betti number of the complex manifold $X(\mathbb{C})$, and $\text{NS}(X)$ denotes the Néron-Severi group of $X \otimes_k \bar{k}$. It is well-known that $b_1(X)$ does not depend on the choice of an embedding $k \hookrightarrow \mathbb{C}$, and $\text{NS}(X)$ is a finitely generated abelian group. We put $\text{NS}(X)_{\mathbb{R}} := \text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. The Albanese variety of X is denoted by $\text{Alb}(X)$.

Outline of this paper. In Section 2, we recall the definitions and some properties of dynamical and arithmetic degrees. In Section 3, we prove some lemmas about reduction of Conjecture 1.1. In Section 4, we study some sufficient conditions for endomorphisms on product varieties to be split. These are important to prove Theorem 1.3 and Theorem 1.4. Theorem 1.3 is proved in Section 5, and Theorem 1.4 is proved in Section 6. Finally, in Section 7, we prove Theorem 1.5.

2. DYNAMICAL DEGREE AND ARITHMETIC DEGREE

Let H be an ample divisor on a smooth projective variety X defined over a number field k . The (first) *dynamical degree* of a dominant rational self-map $f: X \dashrightarrow X$ is defined by

$$\delta_f := \lim_{n \rightarrow \infty} \deg(((f^n)^* H) \cdot H^{\dim X - 1})^{1/n}.$$

The limit defining δ_f exists, and δ_f does not depend on the choice of H (see [6, Corollary 7], [10, Proposition 1.2]). Note that if f is an endomorphism, we have $(f^n)^* = (f^*)^n$ as a linear self-map on $\text{NS}(X)$. But if f is merely a rational self-map, we have $(f^n)^* \neq (f^*)^n$ in general.

Remark 2.1 ([12, Section 3], [21, p. 675]). Let $f: \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ be a dominant rational self-map on the projective space \mathbb{P}^N defined over k . In homogeneous coordinates, f can be written as

$$f: [X_0 : \cdots : X_N] \mapsto [P_0(X_0, \dots, X_N) : \cdots : P_N(X_0, \dots, X_N)],$$

where P_0, \dots, P_N are homogeneous polynomials of the same degree without common nontrivial factors (see [12, Lemma 2.9]). The common degree of these polynomials is written as $\deg(f)$, and we call it the *degree* of f . Then the following equality holds:

$$\delta_f = \lim_{n \rightarrow \infty} (\deg(f^n))^{1/n}.$$

Note that, in general, this notion is different from the degree of the extension of function fields induced by f . The logarithm of the right hand side is called the *algebraic entropy*. For details, see [12], [21] and references therein.

Remark 2.2 ([6, Proposition 1.2 (iii)], [18, Remark 7]). Let $\rho(f^*)$ be the spectral radius of the linear self-map $f^*: \text{NS}(X)_{\mathbb{R}} \rightarrow \text{NS}(X)_{\mathbb{R}}$. The dynamical degree δ_f is equal to the limit $\lim_{n \rightarrow \infty} (\rho((f^n)^*))^{1/n}$. Thus we have $\delta_{f^n} = \delta_f^n$ for every $n \geq 1$.

Let $X_f(\bar{k})$ be the set of \bar{k} -rational points on X at which f^n is defined for every $n \geq 1$. The *arithmetic degree* of f at a \bar{k} -rational point $P \in X_f(\bar{k})$ is defined as follows. Let

$$h_H: X(\bar{k}) \rightarrow [0, \infty)$$

be the (absolute logarithmic) Weil height function associated with H (see [13, Theorem B3.2]). We put

$$h_H^+(P) := \max \{h_H(P), 1\}.$$

We call

$$\begin{aligned} \bar{\alpha}_f(P) &:= \limsup_{n \rightarrow \infty} h_H^+(f^n(P))^{1/n}, \text{ and} \\ \underline{\alpha}_f(P) &:= \liminf_{n \rightarrow \infty} h_H^+(f^n(P))^{1/n}, \end{aligned}$$

the *upper arithmetic degree* and the *lower arithmetic degree*, respectively. It is known that $\bar{\alpha}_f(P)$ and $\underline{\alpha}_f(P)$ do not depend on the choice of H (see [18, Proposition 12]). If $\bar{\alpha}_f(P) = \underline{\alpha}_f(P)$, the limit

$$\alpha_f(P) := \lim_{n \rightarrow \infty} h_H^+(f^n(P))^{1/n}$$

is called the *arithmetic degree of f at P* .

Remark 2.3. When f is an endomorphism, the existence of the limit defining the arithmetic degree $\alpha_f(P)$ is proved by Kawaguchi and Silverman in [16, Theorem 3]. But it is not known in general.

Remark 2.4. The inequality $\bar{\alpha}_f(P) \leq \delta_f$ is proved by Kawaguchi and Silverman in [18, Theorem 4]. Hence, in order to prove Conjecture 1.1, it is enough to prove the opposite inequality $\underline{\alpha}_f(P) \geq \delta_f$. Similarly, in order to prove Conjecture 1.2, it is enough to prove that the set

$$\{P \in X_f(\bar{k}) : \underline{\alpha}_f(P) \geq \delta_f\}$$

contains a Zariski dense set of points having disjoint orbits.

We recall the following result on relative dynamical degrees proved by T.-C. Dinh and V.-A. Nguyen.

Theorem 2.5 ([5, Theorem 1.1]). *Let X and Y be smooth projective varieties defined over \mathbb{C} , with $\dim X \geq \dim Y$. Let $f: X \dashrightarrow X$, $g: Y \dashrightarrow Y$ and $\pi: X \dashrightarrow Y$ be dominant rational maps such that $\pi \circ f = g \circ \pi$. Then we have*

$$d_p(f) = \max_{\max\{0, p - \dim X + \dim Y\} \leq j \leq \min\{p, \dim Y\}} d_j(g) d_{p-j}(f|_{\pi})$$

for every $0 \leq p \leq \dim X$.

Here, $d_p(f)$ and $d_p(f|_{\pi})$ are the p -th dynamical degree and the p -th relative dynamical degree, respectively, defined in [5, Section 3].

Corollary 2.6. Let k be a subfield of \mathbb{C} . Let $f: X \dashrightarrow X$ and $g: Y \dashrightarrow Y$ be dominant rational self-maps on smooth projective varieties of dimension ≥ 1 defined over k . Let $f \times g: X \times Y \dashrightarrow X \times Y$ be the product of f and g . Then we have $\delta_{f \times g} = \max\{\delta_f, \delta_g\}$.

Proof. Since the (first) dynamical degrees do not change when the base field k is extended, we may assume $k = \mathbb{C}$. We apply Theorem 2.5 for $X \times Y$, Y , $f \times g$, g , pr_2 and $p = 1$. We have $d_1(g) = \delta_g$, $d_0(g) = 1$, $d_1((f \times g)|_{\text{pr}_2}) = \delta_f$, $d_0((f \times g)|_{\text{pr}_2}) = 1$, and $d_1(f \times g) = \delta_{f \times g}$ (see [5, Section 3]). Hence we get

$$\begin{aligned} \delta_{f \times g} &= d_1(f \times g) \\ &= \max\{d_1(g)d_0((f \times g)|_{\text{pr}_2}), d_0(g)d_1((f \times g)|_{\text{pr}_2})\} \\ &= \max\{\delta_f, \delta_g\}. \end{aligned}$$

□

3. SOME REDUCTIONS OF THE KAWAGUCHI-SILVERMAN CONJECTURE.

In this section, we prove lemmas which are useful to prove some cases of Conjecture 1.1.

Lemma 3.1. Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive real numbers. Assume that the limits $\lim_{n \rightarrow \infty} a_n^{1/n}$ and $\lim_{n \rightarrow \infty} b_n^{1/n}$ exist and are not less than 1. Then the limit $\lim_{n \rightarrow \infty} (a_n + b_n)^{1/n}$ exists and is equal to $\max\{\lim_{n \rightarrow \infty} a_n^{1/n}, \lim_{n \rightarrow \infty} b_n^{1/n}\}$.

Proof. We put $\alpha := \lim_{n \rightarrow \infty} a_n^{1/n}$ and $\beta := \lim_{n \rightarrow \infty} b_n^{1/n}$. If $\alpha > \beta$, we have

$$a_n^{1/n} \leq (a_n + b_n)^{1/n} \leq (2a_n)^{1/n}$$

for all sufficiently large n . Hence we have $\lim_{n \rightarrow \infty} (a_n + b_n)^{1/n} = \alpha$. The proof for the case $\alpha < \beta$ is similar.

If $\alpha = \beta$, since we have $\lim_{n \rightarrow \infty} (a_n + b_n)^{1/n} = \alpha \cdot \lim_{n \rightarrow \infty} (1 + b_n/a_n)^{1/n}$, it is enough to prove the assertion when $a_n = 1$ for all n and $\beta = 1$. Fix a real number $0 < \varepsilon < 1$. There exists an integer n_0 such that $b_n \leq (1 + \varepsilon)^n$ holds for all $n \geq n_0$. Then we have $1 \leq (1 + b_n)^{1/n} \leq (1 + (1 + \varepsilon)^n)^{1/n} \leq (2 \cdot (1 + \varepsilon)^n)^{1/n} = 2^{1/n}(1 + \varepsilon)$. Hence we get $1 \leq \lim_{n \rightarrow \infty} (1 + b_n)^{1/n} \leq 1 + \varepsilon$, and the assertion follows. □

Lemma 3.2. Let X and Y be smooth projective varieties defined over a number field k . Let $f: X \rightarrow X$ and $g: Y \rightarrow Y$ be dominant endomorphisms defined over k , respectively. Assume that Conjecture 1.1 is true for f and g . Then Conjecture 1.1 is true for the product endomorphism $f \times g: X \times Y \rightarrow X \times Y$.

Proof. Let D_1 and D_2 be ample divisors on X and Y , respectively. Then $H := \text{pr}_1^* D_1 + \text{pr}_2^* D_2$ is an ample divisor on $X \times Y$ (see [11, II. Proposition 7.10]). Fix the Weil height function associated with H , $\text{pr}_1^* D_1$ and $\text{pr}_2^* D_2$ to satisfy that $h_H = h_{\text{pr}_1^* D_1} + h_{\text{pr}_2^* D_2}$. Since D_1 and D_2 are ample, we may assume that $h_{\text{pr}_1^* D_1}$ and $h_{\text{pr}_2^* D_2}$ are positive functions. For every \bar{k} -rational point $P \in (X \times Y)(\bar{k})$ whose forward $(f \times g)$ -orbit $\mathcal{O}_{f \times g}(P)$ is Zariski

dense in $X \times Y$, we have

$$\begin{aligned}
\delta_{f \times g} &= \max\{\delta_f, \delta_g\} && \text{Corollary 2.6} \\
&= \max\{\alpha_f(\text{pr}_1(P)), \alpha_g(\text{pr}_2(P))\} && \text{Conjecture 1.1 is true for } f \text{ and } g \\
&= \lim_{n \rightarrow \infty} (h_{\text{pr}_1^* D_1}^+((f \times g)^n(P)) + h_{\text{pr}_2^* D_2}^+((f \times g)^n(P)))^{1/n} && \text{Lemma 3.1} \\
&= \lim_{n \rightarrow \infty} h_H^+((f \times g)^n(P))^{1/n} \\
&= \alpha_{f \times g}(P).
\end{aligned}$$

Hence Conjecture 1.1 is true for $f \times g$. \square

Lemma 3.3. *Let X be a smooth projective variety defined over a number field k , and $f: X \rightarrow X$ an endomorphism defined over k . Then Conjecture 1.1 is true for f if and only if Conjecture 1.1 is true for f^t for some $t \geq 1$.*

Proof. One direction is trivial. Assume that Conjecture 1.1 is true for f^t for some $t \geq 1$. For every \bar{k} -rational point $P \in X(\bar{k})$, we have

$$\mathcal{O}_f(P) = \bigcup_{i=0}^{t-1} f^i(\mathcal{O}_{f^t}(P)).$$

Therefore, $\mathcal{O}_f(P)$ is Zariski dense in X if and only if $\mathcal{O}_{f^t}(P)$ is Zariski dense in X . Since we know the existence of $\alpha_f(P)$ (see Remark 2.3), we get

$$\begin{aligned}
\alpha_f(P) &= \lim_{n \rightarrow \infty} h_H^+(f^n(P))^{1/n} \\
&= \lim_{n \rightarrow \infty} h_H^+((f^t)^n(P))^{1/tn} \\
&= \alpha_{f^t}(P)^{1/t} \\
&= \delta_{f^t}^{1/t} \\
&= \delta_f && \text{Remark 2.2}
\end{aligned}$$

Hence Conjecture 1.1 is true for f . \square

4. SPLITTING OF ENDOMORPHISMS ON PRODUCT VARIETIES

In this section, we work over \mathbb{C} . All varieties and morphisms are defined over \mathbb{C} .

Let X and Y be smooth projective varieties, and $f: X \times Y \rightarrow X \times Y$ a dominant endomorphism. For a \mathbb{C} -rational point $x \in X(\mathbb{C})$, let $i_x: Y \hookrightarrow X \times Y$ be the closed embedding defined by $i_x(y) := (x, y)$. For a \mathbb{C} -rational point $y \in Y(\mathbb{C})$, let $j_y: X \hookrightarrow X \times Y$ be the closed embedding defined by $j_y(x) := (x, y)$. We put $X_y := j_y(X) \subset X \times Y$.

We say an endomorphism $f: X \times Y \rightarrow X \times Y$ is *split* if there exist endomorphisms $g: X \rightarrow X$ and $h: Y \rightarrow Y$ satisfying $f = g \times h$. In this section, we study sufficient conditions for endomorphisms on product varieties to be split.

4.1. Some lemmas on endomorphisms on product varieties.

Lemma 4.1. *Let $y \in Y(\mathbb{C})$ be a \mathbb{C} -rational point, and $Z \subset X \times Y$ a closed subvariety with $\dim Z = \dim Y$. Then the following are equivalent.*

- $\text{pr}_2|_Z: Z \rightarrow Y$ is dominant.
- $\deg(X_y \cdot Z) \neq 0$. Here, $X_y \cdot Z$ is a zero cycle on $X \times Y$, which is the intersection product of cycles X_y and Z on $X \times Y$.

Proof. If $\text{pr}_2|_Z$ is dominant, we have $\text{pr}_{2,*}(Z) = \deg(Z/Y)Y$ as cycles on Y (see [9, 1.4]). Since $X_y = \text{pr}_2^*(y)$ as cycles on $X \times Y$, we have

$$\begin{aligned} \deg(X_y \cdot Z) &= \deg(\text{pr}_2^*(y) \cdot Z) \\ &= \deg(y \cdot \text{pr}_{2,*}(Z)) \quad \text{by the projection formula ([9, Example 8.1.7])} \\ &= \deg(Z/Y) \\ &\geq 1. \end{aligned}$$

Conversely, if $\text{pr}_2|_Z$ is not dominant, $\text{pr}_2|_Z$ is not surjective. We may assume $y \notin \text{pr}_2(Z)$ because $\deg(X_y \cdot Z)$ does not depend on y ; see Remark 4.2 below. Since $X_y \cap Z = \emptyset$, we have $\deg(X_y \cdot Z) = 0$. \square

Remark 4.2. For $y, y' \in Y(\mathbb{C})$, the cycles X_y and $X_{y'}$ on $X \times Y$ are algebraically equivalent. Hence the degree $\deg(X_y \cdot Z)$ does not depend on y (see [9, Corollary 10.2.2]).

Lemma 4.3. *There is a positive integer $t \geq 1$ such that the endomorphism*

$$\text{pr}_2 \circ f^t \circ i_x : Y \longrightarrow Y$$

is dominant for every $x \in X(\mathbb{C})$.

Proof. Fix \mathbb{C} -rational points $x_0 \in X(\mathbb{C})$ and $y_0 \in Y(\mathbb{C})$. We put $Z_{x_0,t} := f^t(i_{x_0}(Y))$ for each $t \geq 0$. We have $\dim Z_{x_0,t} = \dim Y$ because f is finite (see [2, Lemma 1], [8, Lemma 2.3 (1)]). Let us consider the cohomology classes $\text{cl}(X_{y_0}) \in H^{2\dim Y}(X(\mathbb{C}) \times Y(\mathbb{C}), \mathbb{Q})$ and $\text{cl}(Z_{x_0,t}) \in H^{2\dim X}(X(\mathbb{C}) \times Y(\mathbb{C}), \mathbb{Q})$. Recall that the intersection number is calculated by the cup product of cohomology classes. Hence, we have $\text{cl}(X_{y_0}) \cup \text{cl}(Z_{x_0,t}) = \deg(X_{y_0} \cdot Z_{x_0,t})$ in $H^{2\dim X + 2\dim Y}(X(\mathbb{C}) \times Y(\mathbb{C}), \mathbb{Q}) = \mathbb{Q}$. By Lemma 4.1, $\text{pr}_2 \circ f^t \circ i_{x_0}$ is dominant if and only if $\deg(X_{y_0} \cdot Z_{x_0,t}) \neq 0$. Since $\text{cl}(Z_{x_0,t})$ does not depend on x_0 , we see that $\text{pr}_2 \circ f^t \circ i_x$ is dominant for one $x \in X(\mathbb{C})$ if and only if it is dominant for every $x \in X(\mathbb{C})$. Therefore, it is enough to consider the case of $\text{pr}_2 \circ f^t \circ i_{x_0}$.

Since $H^{2\dim X}(X(\mathbb{C}) \times Y(\mathbb{C}), \mathbb{Q})$ is a finite dimensional \mathbb{Q} -vector space, there is a unique integer $s \geq 0$ such that $\text{cl}(Z_{x_0,i})$ ($0 \leq i \leq s$) are linearly independent over \mathbb{Q} , but $\text{cl}(Z_{x_0,i})$ ($0 \leq i \leq s+1$) are linearly dependent over \mathbb{Q} . We write $\text{cl}(Z_{x_0,s+1}) = \sum_{i=0}^s a_i \text{cl}(Z_{x_0,i})$ for some $a_i \in \mathbb{Q}$.

Assume that $\text{pr}_2 \circ f^t \circ i_{x_0}$ is not dominant for all $t \geq 1$. Then we have $\text{cl}(X_{y_0}) \cup \text{cl}(Z_{x_0,t}) = 0$ for all $t \geq 1$. On the other hand, since $\text{pr}_2 \circ i_{x_0} = \text{id}_Y$, we have $\text{cl}(X_{y_0}) \cup \text{cl}(Z_{x_0,0}) = 1$. Calculating the cup products with $\text{cl}(X_{y_0})$, we have

$$0 = \text{cl}(X_{y_0}) \cup \text{cl}(Z_{x_0,s+1}) = \sum_{i=0}^s a_i (\text{cl}(X_{y_0}) \cup \text{cl}(Z_{x_0,i})) = a_0.$$

If $s = 0$, we get $\text{cl}(Z_{x_0,1}) = 0$. But this is a contradiction because $\text{cl}(Z_{x_0,1})$ is a prime cycle. Thus we may assume $s \geq 1$.

Recall that f induces a bijective \mathbb{Q} -linear map

$$f_* : H^{2\dim X}(X(\mathbb{C}) \times Y(\mathbb{C}), \mathbb{Q}) \xrightarrow{\cong} H^{2\dim X}(X(\mathbb{C}) \times Y(\mathbb{C}), \mathbb{Q})$$

satisfying $f_*(\text{cl}(Z_{x_0,t})) = \text{cl}(f_*(Z_{x_0,t})) = \deg(Z_{x_0,t}/Z_{x_0,t+1}) \cdot \text{cl}(Z_{x_0,t+1})$ (see [2, Lemma 1]). We put

$$\alpha := \sum_{i=1}^s a_i \cdot \deg(Z_{x_0,i-1}/Z_{x_0,i})^{-1} \cdot \text{cl}(Z_{x_0,i-1}).$$

Since $a_0 = 0$, we have

$$\begin{aligned} f_*(\alpha) &= \sum_{i=1}^s a_i \operatorname{cl}(Z_{x_0,i}) \\ &= \operatorname{cl}(Z_{x_0,s+1}) \\ &= f_*(\deg(Z_{x_0,s}/Z_{x_0,s+1})^{-1} \cdot \operatorname{cl}(Z_{x_0,s})). \end{aligned}$$

By the injectivity of f_* , we have $\alpha = \deg(Z_{x_0,s}/Z_{x_0,s+1})^{-1} \cdot \operatorname{cl}(Z_{x_0,s})$. But it contradicts with the assumption that $\operatorname{cl}(Z_{x_0,i})$ ($0 \leq i \leq s$) are linearly independent over \mathbb{Q} . Therefore, $\operatorname{pr}_2 \circ f^t \circ i_{x_0}$ is dominant for some $t \geq 1$. \square

4.2. Splitting of endomorphisms on product varieties (1). For a smooth projective variety X , the automorphism group of X , denoted by $\operatorname{Aut}(X)$, has a natural structure of a group scheme locally of finite type (see [20, Theorem 3.7]). Its neutral component is denoted by $\operatorname{Aut}^\circ(X)$. Let $\operatorname{Sur}(X)$ be the scheme of surjective endomorphisms on X , which has a natural action of $\operatorname{Aut}(X)$ (for details, see [8], [3]).

Lemma 4.4. *Let X be a smooth projective variety.*

- (1) $b_1(X) = 0$ if and only if $\operatorname{Alb}(X) = 0$.
- (2) If $b_1(X) = 0$, $\operatorname{Aut}^\circ(X)$ is a linear algebraic group.

Proof. The part (1) follows from the equality $b_1(X) = 2 \cdot \dim \operatorname{Alb}(X)$ (see [1, Theorem V.13]). For the part (2), see [7, Corollary 5.8], [3, p.73, Remark 2]. \square

Lemma 4.5. *Let X and Y be smooth projective varieties, and $f: X \times Y \rightarrow X \times Y$ a dominant endomorphism. Assume that at least one of the following conditions is satisfied:*

- $b_1(X) = 0$, or
- $\operatorname{Aut}^\circ(Y)$ is a linear algebraic group.

Then there is an integer $t \geq 1$ such that $f^t(x, y) = (g(x, y), h(y))$ for some morphisms $g: X \times Y \rightarrow X$ and $h: Y \rightarrow Y$.

Proof. Take an integer $t \geq 1$ as in Lemma 4.3. Fix a \mathbb{C} -rational point $x_0 \in X(\mathbb{C})$, and get a holomorphic map $X(\mathbb{C}) \rightarrow \operatorname{Sur}(Y)(\mathbb{C}), x \mapsto \operatorname{pr}_2 \circ f^t \circ i_x$. The image of $X(\mathbb{C})$ in $\operatorname{Sur}(Y)(\mathbb{C})$ is contained in a left $\operatorname{Aut}^\circ(Y)(\mathbb{C})$ -orbit by Horst's theorem [14, Theorem 3.1]. Hence there is a holomorphic map $\varphi: X(\mathbb{C}) \rightarrow \operatorname{Aut}^\circ(Y)(\mathbb{C})$ satisfying $\operatorname{pr}_2 \circ f^t \circ i_x = \varphi(x) \circ \operatorname{pr}_2 \circ f^t \circ i_{x_0}$ for all $x \in X(\mathbb{C})$.

We shall show φ is constant. Since $\operatorname{Aut}^\circ(Y)$ is a connected algebraic group, by Chevalley's theorem ([4]), there is a linear normal subgroup $\operatorname{Aut}^\circ(Y)_{\operatorname{lin}} \subset \operatorname{Aut}^\circ(Y)$ such that $\operatorname{Aut}^\circ(Y)/\operatorname{Aut}^\circ(Y)_{\operatorname{lin}}$ is an abelian variety. If $b_1(X) = 0$, $\operatorname{Alb}(X)$ is trivial by Lemma 4.4 (1). By the universal property of the Albanese varieties (see [1, Theorem V.13]), the image of $X(\mathbb{C})$ in $(\operatorname{Aut}^\circ(Y)/\operatorname{Aut}^\circ(Y)_{\operatorname{lin}})(\mathbb{C})$ is trivial. Hence φ is a holomorphic map from $X(\mathbb{C})$ to $(\operatorname{Aut}^\circ(Y)_{\operatorname{lin}})(\mathbb{C})$. Since X is a projective variety and $\operatorname{Aut}^\circ(Y)_{\operatorname{lin}}$ is an affine variety, φ is constant. Similarly, if $\operatorname{Aut}^\circ(Y)$ is a linear algebraic group, we have $\operatorname{Aut}^\circ(Y) = \operatorname{Aut}^\circ(Y)_{\operatorname{lin}}$, and φ is constant.

Therefore, we conclude $\varphi(x) = \operatorname{id}_X$ for all $x \in X(\mathbb{C})$. Putting $g := \operatorname{pr}_1 \circ f^t$ and $h := \operatorname{pr}_2 \circ f^t \circ i_{x_0}$, we have $f^t(x, y) = (g(x, y), h(y))$. \square

Theorem 4.6. *Let X and Y be smooth projective varieties, and $f: X \times Y \rightarrow X \times Y$ a dominant endomorphism. Assume that at least one of the following conditions is satisfied:*

- $b_1(X) = 0$,
- $b_1(Y) = 0$, or
- $\text{Aut}^\circ(X)$ and $\text{Aut}^\circ(Y)$ are linear algebraic groups.

Then f^t is split for some $t \geq 1$.

Proof. By changing the role of X and Y , we may assume that the first condition or the third condition is satisfied. Take an integer $t \geq 1$ as in Lemma 4.5. Fix \mathbb{C} -rational points $x_0 \in X(\mathbb{C})$ and $y_0 \in Y(\mathbb{C})$. We have $f^t(x, y) = (g_1(x, y), h_1(y))$ for some morphisms $g_1: X \times Y \rightarrow X$ and $h_1: Y \rightarrow Y$.

By Lemma 4.1, for each $y \in Y(\mathbb{C})$, the endomorphism $g_1 \circ j_y: X \rightarrow X$ is finite because it is the restriction of f^t to $j_y(X) \subset X \times Y$ and f^t is finite (see [2, Lemma 1], [8, Lemma 2.3 (1)]). Hence $g_1 \circ j_y$ is surjective for each $y \in Y(\mathbb{C})$. We have a holomorphic map $Y(\mathbb{C}) \rightarrow \text{Sur}(X)(\mathbb{C})$, $y \mapsto g_1 \circ j_y$. The image of $Y(\mathbb{C})$ in $\text{Sur}(X)(\mathbb{C})$ is contained in a left $\text{Aut}^\circ(X)(\mathbb{C})$ -orbit by Horst's theorem [14, Theorem 3.1]. We have a holomorphic map $\psi: Y(\mathbb{C}) \rightarrow \text{Aut}^\circ(X)(\mathbb{C})$ satisfying $g_1 \circ j_y = \psi(y) \circ g_1 \circ j_{y_0}$ for all $y \in Y(\mathbb{C})$. If $b_1(X) = 0$, $\text{Aut}^\circ(X)$ is a linear algebraic group by Lemma 4.4 (2). Since Y is a projective variety and $\text{Aut}^\circ(X)$ is an affine variety, ψ is constant. Similarly, if $\text{Aut}^\circ(X)$ is a linear algebraic group, ψ is constant.

Therefore, we conclude $\psi(y) = \text{id}_X$ for all $y \in Y(\mathbb{C})$. Putting $g(x) = g_1(x, y_0)$ and $h(y) = h_1(y)$, we have $f^t = g \times h$. \square

4.3. Splitting of endomorphisms on product varieties (2). In this subsection, we consider smooth projective varieties X and Y satisfying

$$\text{rank NS}(X \times Y) = \text{rank NS}(X) + \text{rank NS}(Y).$$

Under this assumption, we shall prove results similar to Theorem 4.6 for dominant endomorphisms on $X \times Y$.

Lemma 4.7. *Let X and Y be smooth projective varieties satisfying*

$$\text{rank NS}(X \times Y) = \text{rank NS}(X) + \text{rank NS}(Y).$$

Fix \mathbb{C} -rational points $x_0 \in X(\mathbb{C})$ and $y_0 \in Y(\mathbb{C})$.

(1) *The following maps are isomorphisms and inverses to each other:*

$$\text{NS}(X)_{\mathbb{R}} \oplus \text{NS}(Y)_{\mathbb{R}} \xrightarrow{\cong} \text{NS}(X \times Y)_{\mathbb{R}}, \quad (\alpha, \beta) \mapsto \text{pr}_1^* \alpha + \text{pr}_2^* \beta,$$

$$\text{NS}(X \times Y)_{\mathbb{R}} \xrightarrow{\cong} \text{NS}(X)_{\mathbb{R}} \oplus \text{NS}(Y)_{\mathbb{R}}, \quad \gamma \mapsto (j_{y_0}^* \gamma, i_{x_0}^* \gamma).$$

(2) *For a closed subvariety $\iota_Z: Z \hookrightarrow X$ and an element $\alpha \in \text{NS}(X \times Y)_{\mathbb{R}}$, the following are equivalent:*

- $(\iota_Z \times \text{id}_Y)^* \alpha \in \text{NS}(Z \times Y)_{\mathbb{R}}$ is ample.
- $(j_{y_0} \circ \iota_Z)^* \alpha \in \text{NS}(Z)_{\mathbb{R}}$ and $i_{x_0}^* \alpha \in \text{NS}(Y)_{\mathbb{R}}$ are ample.

Proof. The part (1) is clear. For the part (2), by (1), we have $\alpha = \text{pr}_1^* j_{y_0}^* \alpha + \text{pr}_2^* i_{x_0}^* \alpha$. Hence we have

$$\begin{aligned} (\iota_Z \times \text{id}_Y)^* \alpha &= (\text{pr}_1 \circ (\iota_Z \times \text{id}_Y))^* (j_{y_0}^* \alpha) + (\text{pr}_2 \circ (\iota_Z \times \text{id}_Y))^* (i_{x_0}^* \alpha) \\ &= \text{pr}_Z^* ((j_{y_0} \circ \iota_Z)^* \alpha) + \text{pr}_Y^* (i_{x_0}^* \alpha), \end{aligned}$$

where $\text{pr}_Z: Z \times Y \rightarrow Z$ and $\text{pr}_Y: Z \times Y \rightarrow Y$ are projections. Therefore, $(\iota_Z \times \text{id}_Y)^* \alpha$ is ample if and only if $(j_{y_0} \circ \iota_Z)^* \alpha$ and $i_{x_0}^* \alpha$ are ample (see [11, Proposition 7.10]). \square

Lemma 4.8. *Let X and Y be smooth projective varieties satisfying*

$$\text{rank NS}(X \times Y) = \text{rank NS}(X) + \text{rank NS}(Y),$$

and $f: X \times Y \rightarrow X \times Y$ an endomorphism. If $\text{pr}_2 \circ f \circ i_{x_0}: Y \rightarrow Y$ is dominant for some $x_0 \in X(\mathbb{C})$, we have $f(x, y) = (g(x, y), h(y))$ for some morphisms $g: X \times Y \rightarrow X$ and $h: Y \rightarrow Y$.

Proof. It suffices to prove $\text{pr}_2 \circ f \circ j_{y_0}: X \rightarrow Y$ is constant. Assume that it is not constant. Since $\dim \text{pr}_2 \circ f \circ j_{y_0}(X) \geq 1$, there is an irreducible curve $\iota_Z: Z \hookrightarrow X$ with $\dim \text{pr}_2 \circ f \circ j_{y_0}(Z) = 1$. Let $\delta \in \text{NS}(Y)_{\mathbb{R}}$ be the class of an ample divisor on Y . Since $\text{pr}_2 \circ f \circ j_{y_0} \circ \iota_Z: Z \rightarrow Y$ is finite, the pullback

$$(\text{pr}_2 \circ f \circ j_{y_0} \circ \iota_Z)^* \delta = (j_{y_0} \circ \iota_Z)^* (\text{pr}_2 \circ f)^* \delta \in \text{NS}(Z)_{\mathbb{R}}$$

is ample. On the other hand, since $\text{pr}_2 \circ f \circ i_{x_0}$ is a dominant endomorphism on Y , it is finite (see [2, Lemma 1], [8, Lemma 2.3 (1)]), and the pullback

$$(\text{pr}_2 \circ f \circ i_{x_0})^* \delta = (i_{x_0})^* (\text{pr}_2 \circ f)^* \delta \in \text{NS}(Y)_{\mathbb{R}}$$

is ample. Applying Lemma 4.7 (2) for $\alpha = (\text{pr}_2 \circ f)^* \delta$, we see that

$$(\iota_Z \times \text{id}_Y)^* (\text{pr}_2 \circ f)^* \delta = (\text{pr}_2 \circ f \circ (\iota_Z \times \text{id}_Y))^* \delta \in \text{NS}(Z \times Y)_{\mathbb{R}}$$

is ample. Since the pullback of δ by $\text{pr}_2 \circ f \circ (\iota_Z \times \text{id}_Y)$ is ample, the morphism $\text{pr}_2 \circ f \circ (\iota_Z \times \text{id}_Y): Z \times Y \rightarrow Y$ must be finite. It is a contradiction because $\dim(Z \times Y) > \dim Y$. Hence $\text{pr}_2 \circ f \circ j_{y_0}$ is constant as required. \square

Theorem 4.9. *Let X and Y be smooth projective varieties satisfying*

$$\text{rank NS}(X \times Y) = \text{rank NS}(X) + \text{rank NS}(Y)$$

and $f: X \times Y \rightarrow X \times Y$ an endomorphism. Then f^t is split for some $t \geq 1$.

Proof. Applying Lemma 4.3 and Lemma 4.8, there is an integer $t_1 \geq 1$ such that $f^{t_1}(x, y) = (g_1(x, y), h_1(y))$ for some morphisms $g_1: X \times Y \rightarrow X$ and $h_1: Y \rightarrow Y$. Changing the role of X and Y and applying Lemma 4.3 and Lemma 4.8 again, there is an integer $t_2 \geq 1$ such that $f^{t_1 t_2}(x, y) = (g_2(x), h_2(x, y))$ for some morphisms $g_2: X \rightarrow X$ and $h_2: X \times Y \rightarrow Y$. Then we have $f^{t_1 t_2}(x, y) = (g_2(x), h_1^{t_2}(y))$. Hence $f^{t_1 t_2}$ is split. \square

5. PROOF OF THEOREM 1.3

The base field k is a number field in this section.

For a smooth projective variety X defined over k , we say “Conjecture 1.1 is true for endomorphisms on X ” if Conjecture 1.1 is true for every dominant endomorphism $f: X \rightarrow X$ defined over a finite extension of k and every \bar{k} -rational point $P \in X(\bar{k})$ with Zariski dense forward f -orbit.

Lemma 5.1. *Let X and Y be smooth projective varieties defined over k , and $f: X \times Y \rightarrow X \times Y$ an endomorphism defined over k . Fix an embedding $k \hookrightarrow \mathbb{C}$. Assume that f is split over \mathbb{C} , i.e., there exist endomorphisms $g: X \rightarrow X$ and $h: Y \rightarrow Y$ defined over \mathbb{C} satisfying $f = g \times h$. Then, there is a finite extension k'/k such that f is split over k' .*

Proof. Take \bar{k} -rational points $x_0 \in X(\bar{k})$ and $y_0 \in Y(\bar{k})$. We put $g_0 := \text{pr}_1 \circ f \circ j_{y_0}$ and $h_0 = \text{pr}_2 \circ f \circ i_{x_0}$. These are endomorphisms on X and Y , respectively, defined over \bar{k} . By assumption, $f(x, y) = (g_0(x), h_0(y))$ for all $x \in X(\mathbb{C})$ and $y \in Y(\mathbb{C})$. Hence

$f = g_0 \times h_0$ as endomorphisms defined over \bar{k} . Since g and h are defined over a finite extension of k , the assertion follows. \square

Lemma 5.2. *Let X and Y be smooth projective varieties defined over k . Assume that Conjecture 1.1 is true for endomorphisms on X and Y . Moreover, assume that at least one of the following conditions is satisfied:*

- $b_1(X) = 0$,
- $b_1(Y) = 0$,
- $\text{Aut}^0(X)$ and $\text{Aut}^0(Y)$ are linear algebraic groups, or
- $\text{rank NS}(X \times Y) = \text{rank NS}(X) + \text{rank NS}(Y)$.

Then Conjecture 1.1 is true for endomorphisms on $X \times Y$. Moreover if $b_1(X) = b_1(Y) = 0$, we have $b_1(X \times Y) = 0$.

Proof. Let $f: X \times Y \rightarrow X \times Y$ be an endomorphism defined over k . If one of the first, second, or third condition is satisfied, we apply Theorem 4.6 to conclude that f^t is split for some $t \geq 1$. If the fourth condition is satisfied, we apply Theorem 4.9 to conclude that f^t is split over \mathbb{C} for some $t \geq 1$. In any of these cases, by Lemma 5.1, we have $f^t = g \times h$ for endomorphisms $g: X \rightarrow X$ and $h: Y \rightarrow Y$ defined over a finite extension k' of k .

Since Conjecture 1.1 is true for g and h , it is true for f^t by Lemma 3.2. Therefore, it is true for f by Lemma 3.3.

Finally, the assertion on the first Betti number follows from the equality $b_1(X \times Y) = b_1(X) + b_1(Y)$. \square

Lemma 5.3. *Let X be an Enriques or a K3 surface defined over k . Then $b_1(X) = 0$ and Conjecture 1.1 is true for endomorphisms on X .*

Proof. It is well-known that the first Betti number of an Enriques or a K3 surface is zero (see [1, Theorem VIII. 2]). It is well-known that any dominant endomorphism on an Enriques or a K3 surface is an automorphism ([8, Corollary 2.4]). Since Conjecture 1.1 is true for automorphisms on smooth projective surfaces ([17, Theorem 2 (c)]), Conjecture 1.1 is true for endomorphisms on an Enriques or a K3 surface. \square

Now, we shall complete the proof of Theorem 1.3.

Proof of Theorem 1.3. Recall that each X_i satisfies at least one of the following conditions:

- $b_1(X_i) = 0$ and $\text{rank NS}(X_i) = 1$,
- X_i is an abelian variety,
- X_i is an Enriques surface, or
- X_i is a K3 surface.

By Lemma 5.3, there is a unique subset $I \subset \{1, \dots, n\}$ such that X_i is an abelian variety for every $i \in I$, and $b_1(X_i) = 0$ for every $i \notin I$. We put $Y := \prod_{i \in I} X_i$ and $Z = \prod_{i \notin I} X_i$. Then Y is an abelian variety of dimension $\sum_{i \in I} \dim X_i$.

Conjecture 1.1 is true for endomorphisms on Y by [16, Corollary 31], [22, Theorem 2]. On the other hand, by [17, Theorem 2 (a)] and Lemma 5.2, Conjecture 1.1 is true for endomorphisms on Z . We also have $b_1(Z) = 0$.

By Lemma 5.2, Conjecture 1.1 is true for endomorphisms on $Y \times Z$. \square

6. PROOF OF THEOREM 1.4

Proof of Theorem 1.4. We may assume $k = \mathbb{C}$ and Y is of general type. Then the automorphism group $\text{Aut}(Y)$ is finite ([19, Corollari 2], [15, Theorem 11.12]). Hence the neutral component $\text{Aut}^\circ(Y)$ is trivial.

Applying Lemma 4.5, there is an integer $t \geq 1$ such that $f^t(x, y) = (g(x, y), h(y))$ for some morphisms $g: X \times Y \rightarrow X$ and $h: Y \rightarrow Y$.

Since Y is of general type, h is an automorphism ([8, Proposition 2.6]). Replacing t by $t \cdot |\text{Aut}(Y)|$, we may assume $h = \text{id}_Y$.

Then, for every \mathbb{C} -rational point $P \in (X \times Y)(\mathbb{C})$, we have $\text{pr}_2(P) = \text{pr}_2(f^t(P))$. The forward f -orbit $\mathcal{O}_f(P)$ is contained in the set

$$\{Q \in (X \times Y)(\mathbb{C}) : \text{pr}_2(Q) = \text{pr}_2(f^j(P)) \text{ for some } 0 \leq j < t\},$$

which is not Zariski dense in $X \times Y$. □

7. PROOF OF THEOREM 1.5

7.1. Strategy of the proof of Theorem 1.5. The following useful result is proved by Kawaguchi and Silverman in [17], [18].

Lemma 7.1. *Let $f: X \dashrightarrow X$ be a dominant rational self-map on a smooth projective variety X defined over a number field k , and D a divisor on X . For a \bar{k} -rational point $P \in X_f(\bar{k})$ with*

$$\widehat{h}_{f,D}(P) := \liminf_{n \rightarrow \infty} \delta_f^{-n} h_D^+(f^n(P)) > 0,$$

the arithmetic degree $\alpha_f(P)$ is defined and equal to δ_f .

Proof. See [17, Proposition 14] and [18, Theorem 4]. See also Remark 2.4. □

In the following, let $f: \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ be a rational self-map defined over a number field k as in Theorem 1.5. Our goal is to apply Lemma 7.1 for \bar{k} -rational points in a suitable \mathfrak{p} -adic open set of $\mathbb{A}^N(\bar{k})$ for a finite place \mathfrak{p} of k . The proof of Theorem 1.5 has three steps.

- (1) First, we calculate the dynamical degree of f . It is an easy exercise in linear algebra.
- (2) Second, we find a finite place \mathfrak{p} and a \mathfrak{p} -adic open subset $U \subset \mathbb{A}^N(\bar{k})$ satisfying
 - U is stable by f , i.e., for every \bar{k} -rational point $P \in U$, we have $f(P) \in U$, and
 - every \bar{k} -rational point $P = (x_1, x_2, \dots, x_N) \in U$ satisfies $|x_1^{(1)}|_{\mathfrak{p}} \geq |x_1|_{\mathfrak{p}}^{\delta_f}$, where we put $f^n(P) = (x_1^{(n)}, x_2^{(n)}, \dots, x_N^{(n)})$ for $n \geq 1$.
 - $U \setminus f(U) \subset \mathbb{A}^N(\bar{k})$ is a non-empty \mathfrak{p} -adic open set.
- (3) Finally, we prove $\widehat{h}_{f,H}(P) > 0$, where $H \subset \mathbb{P}^N$ is a hyperplane defined over k . Applying Lemma 7.1, we conclude that, for every \bar{k} -rational point $P \in U$, the arithmetic degree $\alpha_f(P)$ is defined and equal to δ_f .

7.2. Dynamical degrees in the situation of Theorem 1.5.

Definition 7.2. Let $\text{Deg}(f)$ be the square matrix of size N whose (i, j) -th entry is equal to $\deg_{x_i} f_j$. We call it the *degree matrix* of f . Concretely, we have

$$\text{Deg}(f) := \begin{pmatrix} \deg_{x_1} f_1 & 0 & 0 & \dots & 0 \\ \deg_{x_2} f_1 & \deg_{x_2} f_2 & 0 & \dots & 0 \\ \deg_{x_3} f_1 & \deg_{x_3} f_2 & \deg_{x_3} f_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \deg_{x_N} f_1 & \deg_{x_N} f_2 & \deg_{x_N} f_3 & \dots & \deg_{x_N} f_N \end{pmatrix}.$$

The (i, j) -th entry of $\text{Deg}(f)$ is denoted by $d_{i,j}^{(1)}$. Similarly, the degree matrix of f^n is denoted by $\text{Deg}(f^n)$, and the (i, j) -th entry of $\text{Deg}(f^n)$ is denoted by $d_{i,j}^{(n)}$.

Lemma 7.3. Let $g: \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ be a dominant rational self-map satisfying the same conditions as f in Theorem 1.5. Let $\text{Deg}(g) = (d'_{i,j})$ be the degree matrix of g . Then the following inequalities hold;

$$\begin{pmatrix} d_{1,1}d'_{1,1} & 0 & 0 & \dots & 0 \\ 0 & d_{2,2}d'_{2,2} & 0 & \dots & 0 \\ 0 & 0 & d_{3,3}d'_{3,3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d_{N,N}d'_{N,N} \end{pmatrix} \leq \text{Deg}(f \circ g) \leq (\text{Deg}(g))(\text{Deg}(f)).$$

Here we use the following notation; for real matrices $A = (a_{i,j})$ and $B = (b_{i,j})$, the inequality $A \leq B$ means $a_{i,j} \leq b_{i,j}$ for all i, j .

Proof. The second inequality is obvious because the entries of $(\text{Deg}(g))(\text{Deg}(f))$ are the largest possible degrees of the monomials which can appear in the polynomials $(f \circ g)_j$, where we put

$$(f \circ g)(x) = ((f \circ g)_1(x), (f \circ g)_2(x), \dots, (f \circ g)_N(x)).$$

Let us prove the first inequality. Since g is dominant, the polynomials g_1, g_2, \dots, g_N are algebraically independent over k . We denote

$$\begin{aligned} f_1 &= F_1(x_2, \dots, x_N)x_1^{d_{1,1}} + (\text{lower terms in } x_1), \\ g_1 &= G_1(x_2, \dots, x_N)x_1^{d'_{1,1}} + (\text{lower terms in } x_1) \end{aligned}$$

for nonzero polynomials F_1, G_1 . We calculate

$$\begin{aligned} (f \circ g)_1 &= F_1(g_2, \dots, g_N)g_1^{d_{1,1}} + (\text{other terms}) \\ &= F_1(g_2, \dots, g_N)G_1(x_2, \dots, x_N)^{d_{1,1}}x_1^{d_{1,1}d'_{1,1}} + (\text{other terms}). \end{aligned}$$

Since g_2, \dots, g_N are algebraically independent over k , we have $F_1(g_2, \dots, g_N) \neq 0$. Consequently, we have $d_{1,1}d'_{1,1} = (\text{Deg}(f \circ g))_{1,1}$. We can prove $d_{i,i}d'_{i,i} = (\text{Deg}(f \circ g))_{i,i}$ for $i = 2, 3, \dots, N$ by the same way. \square

The proof of the following lemma is omitted because it is an easy exercise in linear algebra.

Lemma 7.4. Let $A \in M_m(\mathbb{C})$ be a complex square matrix of size m . Let $a_{i,j}^{(n)}$ be the (i, j) -th entry of A^n . Then we have

$$\lim_{n \rightarrow \infty} \max_{1 \leq i, j \leq m} |a_{i,j}^{(n)}|^{1/n} = \rho(A),$$

where $\rho(A)$ denotes the spectral radius of A .

Proposition 7.5. *In the situation of Theorem 1.5, we have $\delta_f = \max_{1 \leq i \leq N} d_{i,i}$.*

Proof. We put $(\text{Deg}(f))^n = (e_{i,j}^{(n)})_{i,j}$. Then we have

$$\begin{aligned} \max_{1 \leq i \leq N} d_{i,i}^n &\leq \deg(f^n) \\ &\leq \max_{1 \leq j \leq N} \sum_{i=1}^N d_{i,j}^{(n)} \\ &\leq \max_{1 \leq j \leq N} \sum_{i=1}^N e_{i,j}^{(n)} \end{aligned}$$

by Lemma 7.3. Taking the n -th root and letting $n \rightarrow \infty$, we have

$$\begin{aligned} \max_{1 \leq i \leq N} d_{i,i} &\leq \delta_f \\ &= \lim_{n \rightarrow \infty} (\deg(f^n))^{1/n} && \text{Remark 2.1} \\ &\leq \max_{1 \leq j \leq N} \max_{1 \leq i \leq N} \lim_{n \rightarrow \infty} (e_{i,j}^{(n)})^{1/n} \\ &= \rho(\text{Deg}(f)) && \text{Lemma 7.4} \\ &= \max_{1 \leq i \leq N} d_{i,i}. \end{aligned}$$

Hence we have $\delta_f = \max_{1 \leq i \leq N} d_{i,i}$. □

7.3. Proof of Theorem 1.5 in the first case. We fix a finite place \mathfrak{p} such that all the coefficients of f_i are \mathfrak{p} -adic units. We also fix an embedding $\bar{k} \hookrightarrow \bar{k}_{\mathfrak{p}}$, and consider the \mathfrak{p} -adic absolute value $|\cdot|_{\mathfrak{p}}$ on \bar{k} . To prove Theorem 1.5 in the first case, it is sufficient to prove the following assertion; if $d_{i,i} > d_{i+1,i+1}$ for each i , there exists a \mathfrak{p} -adic open subset $U \subset X_f(\bar{k})$ such that $\widehat{h}_{f,H}(P) > 0$ for every \bar{k} -rational point $P \in U$.

We fix a constant C with $C > N \cdot \max_{1 \leq i,j \leq N} \deg_{x_i} f_j$. We define a \mathfrak{p} -adic open subset $U \subset \mathbb{A}^N(\bar{k})$ by

$$U := \{(x_1, x_2, \dots, x_N) \in \mathbb{A}^N(\bar{k}) : |x_i|_{\mathfrak{p}} > |x_{i+1}|_{\mathfrak{p}}^C > 1 \text{ for } 1 \leq i \leq N-1\}.$$

For each i , let $u_i x_i^{d_{i,i}} \prod_{l=i+1}^N x_l^{e_{i,l}}$ be the nonzero monomial in f_i which has the largest index with respect to the lexicographic order.

Lemma 7.6. *For every \bar{k} -rational point $P = (x_1, x_2, \dots, x_N) \in U$, we have*

$$\begin{aligned} |x_i^{(1)}|_{\mathfrak{p}} &= |x_i^{d_{i,i}}|_{\mathfrak{p}} \cdot \prod_{l=i+1}^N |x_l^{e_{i,l}}|_{\mathfrak{p}}, \\ |x_1^{(1)}|_{\mathfrak{p}} &= \max_{1 \leq i \leq N} |x_i^{(1)}|_{\mathfrak{p}}. \end{aligned}$$

Moreover, the \mathfrak{p} -adic open subset $U \subset \mathbb{A}^N(\bar{k})$ is stable by f , i.e., for every \bar{k} -rational point $P \in U$, we have $f(P) \in U$.

Proof. First, let $u' \prod_{l=i}^N x_l^{t_l}$ be any nonzero monomial in f_i which is different from $u_i x_i^{d_{i,i}} \prod_{l=i+1}^N x_l^{e_{i,l}}$. We put $e_{i,i} := d_{i,i}$. Let j be the smallest index with $t_j \neq e_{i,j}$. Then

the following inequalities hold.

$$\begin{aligned}
|u_i x_i^{d_{i,i}}|_{\mathfrak{p}} \cdot \prod_{l=i+1}^N |x_l^{e_{i,l}}|_{\mathfrak{p}} &\geq |x_i^{d_{i,i}}|_{\mathfrak{p}} \cdot \prod_{l=i+1}^j |x_l^{e_{i,l}}|_{\mathfrak{p}} && \text{because } u_i \text{ is a } \mathfrak{p}\text{-adic unit} \\
&\geq \prod_{l=i}^{j-1} |x_l^{t_l}|_{\mathfrak{p}} \cdot |x_j^{t_j+1}|_{\mathfrak{p}} && \text{choice of } j \\
&= \prod_{l=i}^j |x_l^{t_l}|_{\mathfrak{p}} \cdot |x_j^{(N-j)/(N-j)}|_{\mathfrak{p}} \\
&> \prod_{l=i}^j |x_l^{t_l}|_{\mathfrak{p}} \cdot \prod_{l=j+1}^N |x_l^{C^{l-j}/(N-j)}|_{\mathfrak{p}} && \text{the definition of } U \\
&> \prod_{l=i}^N |x_l^{t_l}|_{\mathfrak{p}} && C \text{ is sufficiently large.}
\end{aligned}$$

The above inequalities show that the \mathfrak{p} -adic absolute value of $u_i x_i^{d_{i,i}} \prod_{l=i+1}^N x_l^{e_{i,l}}$ is greater than the \mathfrak{p} -adic absolute values of the other monomials in f_i . Hence we have $|x_i^{(1)}|_{\mathfrak{p}} = |x_i^{d_{i,i}}|_{\mathfrak{p}} \cdot \prod_{l=i+1}^N |x_l^{e_{i,l}}|_{\mathfrak{p}}$.

Second, the following inequalities hold.

$$\begin{aligned}
|x_i^{(1)}|_{\mathfrak{p}} &= |x_i^{d_{i,i}}|_{\mathfrak{p}} \cdot \prod_{l=i+1}^N |x_l^{e_{i,l}}|_{\mathfrak{p}} \\
&\geq |x_i^{d_{i,i}}|_{\mathfrak{p}} \\
&= |x_i^{d_{i,i}-1} x_i|_{\mathfrak{p}} \\
&> |x_{i+1}^{C(d_{i,i}-1)} x_i^{(N-i-1)/(N-i-1)}|_{\mathfrak{p}} \\
&\geq |x_{i+1}^{C d_{i+1,i+1}}|_{\mathfrak{p}} \cdot \prod_{l=i+2}^N |x_l^{C^{l-i}/(N-i-1)}|_{\mathfrak{p}} && d_{i,i} - 1 \geq d_{i+1,i+1} \\
&> |x_{i+1}^{(1)}|_{\mathfrak{p}}^C && C^2/(N-i-1) > C e_{i+1,l}.
\end{aligned}$$

Hence we have $f(P) = (x_1^{(1)}, x_2^{(1)}, \dots, x_N^{(1)}) \in U$. Consequently, the subset $U \subset \mathbb{A}^N(\bar{k})$ is stable by f , and $|x_1^{(1)}|_{\mathfrak{p}} = \max_{1 \leq i \leq N} |x_i^{(1)}|_{\mathfrak{p}}$. \square

By the construction of U , it is clear that the set $U \setminus f(U) \subset \mathbb{A}^N(\bar{k})$ is a non-empty \mathfrak{p} -adic open set.

Let us return to the proof of Theorem 1.5 in the first case. For each \bar{k} -rational point $P \in U$, the following inequalities hold.

$$\begin{aligned}
\widehat{h}_{f,H}(P) &= \liminf_{n \rightarrow \infty} \delta_f^{-n} h_H^+(f^n(P)) \\
&\geq \liminf_{n \rightarrow \infty} \max_{1 \leq i \leq N} \delta_f^{-n} \log |x_i^{(n)}|_{\mathfrak{p}} \\
&\geq \liminf_{n \rightarrow \infty} \delta_f^{-n} \log |x_1|_{\mathfrak{p}}^{d_{1,1}^n} \\
&= \liminf_{n \rightarrow \infty} d_{1,1}^{-n} \log |x_1|_{\mathfrak{p}}^{d_{1,1}^n} && \text{because } \delta_f = d_{1,1}
\end{aligned}$$

$$\begin{aligned}
&= \log |x_1|_{\mathfrak{p}} \\
&> 0.
\end{aligned}$$

Applying Lemma 7.1, the proof of Theorem 1.5 in the first case is completed. \square

7.4. Proof of Theorem 1.5 in the second case. Finally, we prove Theorem 1.5 in the second case, i.e., $N = 2$.

If $d_{1,1} > d_{2,2}$, this is the case $N = 2$ of the first case of Theorem 1.5.

If $d_{1,1} \leq d_{2,2}$, we take a finite place \mathfrak{p} such that all the coefficients of f_1 and f_2 are \mathfrak{p} -adic units, and define a \mathfrak{p} -adic open subset $U \subset \mathbb{A}^2(\overline{k})$ by

$$U := \{(x, y) \in \mathbb{A}^2(\overline{k}) : |x_2|_{\mathfrak{p}} > 1\}.$$

It is easy to check the following assertions:

- $\delta_f = d_{2,2}$,
- U is stable by f ,
- $U \setminus f(U)$ is a non-empty \mathfrak{p} -adic open set, and
- $|x_2^{(n)}|_{\mathfrak{p}} = |x_2|_{\mathfrak{p}}^{d_{2,2}^n}$

For every \overline{k} -rational point $P \in U$, the following inequalities hold.

$$\begin{aligned}
\widehat{h}_{f,H}(P) &= \liminf_{n \rightarrow \infty} \delta_f^{-n} h_H^+(f^n(P)) \\
&\geq \liminf_{n \rightarrow \infty} \max_{i=1,2} \delta_f^{-n} \log |x_i^{(n)}|_{\mathfrak{p}} \\
&\geq \liminf_{n \rightarrow \infty} \delta_f^{-n} \log |x_2^{(n)}|_{\mathfrak{p}} \\
&\geq \liminf_{n \rightarrow \infty} d_{2,2}^{-n} \log |x_2|_{\mathfrak{p}}^{d_{2,2}^n} \\
&= \log |x_2|_{\mathfrak{p}} \\
&> 0.
\end{aligned}$$

Applying Lemma 7.1, the proof of Theorem 1.5 in the second case is complete. \square

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